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Matrix theory, U-duality and toroidal compactifications of M-theory

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Abstract: Using U-duality, the properties of the matrix theories corresponding to the compactification of M-theory on T^d are investigated. The couplings of the $d+1$ dimensional effective Super-Yang-Mills theory to all the M-theory moduli is deduced and the spectrum of BPS branes in the SYM gives the corresponding spectrum of the matrix theory. Known results are recovered for $d \leq 5$ and predictions for $d > 5$ are proposed. For $d > 3$, the spectrum includes $d-4$ branes arising from YM instantons, and U-duality interchanges momentum modes with brane wrapping modes. For $d = 6$, there is a generalised θ-angle which couples to instantonic 3-branes and which combines with the SYM coupling constant to take values in $SL(2,\mathbb{R})/U(1)$, acted on by an $SL(2,\mathbb{Z})$ subgroup of the U-duality group $E_6(\mathbb{Z})$. For $d = 4, 7, 8$, there is an $SL(d+1)$ symmetry, suggesting that the matrix theory could be a scale-invariant $d{+}2$ dimensional theory on $T^{d+1} \times \mathbb{R}$ in these cases, as is already known to be the case for $d = 4$; evidence is found suggesting this happens for $d = 8$ but not $d = 7$.

Keywords: M(atrix) Theories, M-Theory, String Duality.

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Contents

1. Matrices and branes

M-theory in the infinite momentum frame is conjectured to be described by the large N limit of the $U(N)$ matrix quantum mechanics obtained from reducing 10-D super Yang-Mills to one dimension [1], [2]. M-theory compactified on T^d for $d \leq 3$ is conjectured to be described by a $d+1$ dimensional matrix theory given by $d+1$ dimensional Super-Yang-Mills (SYM) on $\tilde{T}^d \times \mathbb{R}$ where $\tilde{T}^d$ is the dual torus [1]–[17]. For $d > 3$, the matrix description is given approximately at low energies by $d+1$ dimensional SYM, but as the theory is non-renormalizable, extra degrees of freedom must become important at short distances. These should give rise to a consistent quantum matter theory (i.e. one without gravity) and this has been shown to be the case for $d \leq 5$. For $d = 4$, this is the $5+1$ dimensional (2,0) supersymmetric self-dual tensor multiplets with $U(N)$

gauge symmetry [9]–[12], and for $d = 5$ it is the (2,0) supersymmetric non-critical self-dual string theory [8]–[12]. In each of the cases with $d \leq 5$, the U-duality group is manifest in the matrix description. U-duality then defines much of the structure of the matrix theory, and we will here investigate theories which have an effective description in terms of SYM on $\tilde{T}^d \times \mathbb{R}$ and which are invariant under the appropriate U-duality group. Recently a matrix theory for $d = 6$ has been discussed [15], and some difficulties pointed out [16], [17]. In [3] it was conjectured that M-theory on T^{d+1}, after an infinite boost so that one of the circles becomes null, should be described, in a suitable limit, by a similar matrix theory whose low-energy limit is $d+1$ dimensional SYM on $\tilde{T}^d \times \mathbb{R}$ with $U(N)$ gauge group, for *finite* N. A derivation of this, and of the conjecture of [1], has been proposed in [16], [17].

If a matrix theory exists for $d \geq 6$, then it presumably requires the existence of some 'exotic' matter theory with $U(N)$ gauge symmetry which (i) reduces to $d+1$ dimensional SYM at low energies (ii) reduces to a non-critical string theory in the limit in which T^d decompactifies to $T^5 \times \mathbb{R}^{d-5}$, so that it should contain extended objects (iii) has 'manifest' U-duality under the U-duality group $E_d(\mathbb{Z})$ (where $E_5 = SO(5,5)$, $E_4 = SL(5)$, $E_3 = SL(3) \times SL(2)$ and $E_2 = SL(2) \times \mathbb{R}$). The conjecture of [3] suggests that for finite N, in addition the SYM on $T^d \times \mathbb{R}$ should satisfy the condition (iv) that it should be invariant under $E_{d+1}(\mathbb{Z})$, the duality group for compactification on T^{d+1}. Indeed, in [13] it was suggested that for 4-dimensional SYM on $T^3 \times \mathbb{R}$ the expected $SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$ duality symmetry (the $SL(3,\mathbb{Z})$ symmetry of T^3 and the S-duality of 3+1 dimensional SYM) could be extended to $SL(5,\mathbb{Z})$ by Nahm-type transformations mixing the rank N with electric and magnetic fluxes. It was shown that the BPS spectrum indeed fits into $SL(5,\mathbb{Z})$ representations. In [18], evidence for this will be presented for $d > 3$; the analysis of [14] will be generalised to show that a large class of BPS states, shown to fit into representations of $E_d(Z)$ in [14], in fact fit into representations of $E_{d+1}(Z)$. In [19], it will be shown that the U-duality group for toroidal compactifications of M-theory are unchanged in the limit in which one of the circles becomes null. The properties of the effective SYM theory give a great deal of information about the full matrix theory, much in the same way that the effective supergravity theories give information about non-perturbative string theories [21], and analysing these is the aim of this paper. In particular, BPS solitonic branes of the SYM theory imply the existence of BPS states of the matrix theory which can be extrapolated to BPS states in regimes where the SYM is not a good effective description, just as solitonic branes of supergravity correspond to fundamental strings, D-branes and solitonic branes of non-perturbative string theory [21].

The same $U(N)$ SYM theories in $d+1$ dimensions also emerge in the effective description of N coincident Dd-branes (i.e. D-branes with $d+1$ dimensional world-volume). This is of course not an accident; the matrix theory emerges from the description of N D-branes in a limit in which (at least for low enough d) the bulk fields and the massive string modes decouple [1], [16], [17]. The uncompactified M-theory is formulated in terms of N 0-branes in the limit $N \to \infty$ [1] and this is also a reasonable approximate description for M-theory on T^d if the torus is large. However, this description is missing the string winding modes which become light if the torus is small. For a small torus, T-duality can be used to transform N 0-branes on T^d to N Dd-branes wrapped around the dual torus $\tilde{T}^d$. The perturbative dynamics of the D-branes is given in terms of fundamental strings ending on the D-branes, and the low-energy effective action arising from this for N Dd branes is $d+1$ dimensional $U(N)$ SYM. For $d \leq 3$ the SYM is a well-defined quantum theory and provides a good description of the D-brane dynamics at energies much less than the string scale. For $d \geq 4$, however, SYM must be supplemented by other degrees of freedom. For example, the D4-brane of the IIA theory at finite string coupling is given by the M5-brane wrapped around the circular dimension of M-theory compactified on S^1. The dynamics are then given in terms of the effective world-volume theory of the M5-brane, which is the 5+1-dimensional (2,0) supersymmetric self-dual tensor theory.

In each case, for low enough d, there is a limit in which the string length becomes infinite, supergravity and the tower of massive string states decouple leaving a matter theory with a complete quantum description. As Dd-branes with different values of d are all related by T-duality, it is natural to expect such a limit for all values of d, so that there should be a consistent quantum matter theory for $d \geq 5$ that reduces to $U(N)$ SYM at low energy. By T-duality, it would be this limiting theory, in the additional limit $N \to \infty$, that should be equivalent to M-theory on T^d in the infinite momentum frame. However, already for $d = 6$ there are problems in implementing this procedure – in the limit of the D6-brane considered in [16], [17], it appears that there may be difficulties in decoupling gravity. For $d > 6$, there are further problems in even considering systems of N D-branes: one cannot have arbitrary numbers of D7-branes, the D8-brane requires the modified type IIA theory with a mass term [24] and D9-brane backgrounds are inconsistent except in an orientifold construction, in which case the number of D9-branes is fixed.

For low d, then, the matrix theory for M-theory on T^d is given by a limit of the world-volume theory of N Dd-branes wrapped around T^d. For $d \leq 3$ this gives $d+1$ dimensional SYM on $\tilde{T} \times \mathbb{R}$. For $d = 4$, the relation between the D4-brane and the M5-brane leads to the effective world-volume dynamics of the M5-brane, namely the 5+1 dimensional (2,0) supersymmetric self-dual tensor theory on $T^5 \times \mathbb{R}$. The type IIA string coupling is related to the size of the 11th compactified dimension of M-theory, while the 4+1 dimensional SYM coupling constant is related to the size of the extra dimension for the 6-dimensional tensor theory compactified on a circle. Thus on taking 4+1 dimensional SYM to strong coupling, a tower of states become light; these can be interpreted as Kaluza-Klein modes for a 5+1 dimensional theory compactified on a circle which becomes large as the coupling does. In this case, the states becoming light are the solitons in $4+1$ dimensions arising from Yang-Mills instantons in 4 Euclidean dimensions [9]–[12].

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For $d = 5$, the D5-brane is related by $SL(2, \mathbb{Z})$ duality to the (solitonic) NS 5-brane (or F5-brane) of type IIB, the world-volume theory for which is the (1,1) supersymmetric non-critical string proposed in [12] whose low energy effective theory is 5+1 dimensional SYM. For $d = 5$, the matrix theory is then (1,1) string theory on $T^5 \times \mathbb{R}$, which is T-dual to the (2,0) string theory on $T^5 \times \mathbb{R}$. As the M-theory torus decompactifies from T^5 to $T^4 \times \mathbb{R}$, the tension of the (2,0) string becomes infinite so that it reduces to the (2,0) tensor field theory, as required.

For $d = 6$, the D6-brane arises from the Kaluza-Klein monopole of M-theory (i.e. the solution $\mathbb{R}^{6,1} \times N$ where N is self-dual Taub-NUT space) [22] which can be interpreted as a 6-brane of M-theory [23]. The corresponding matrix theory is then the world-volume theory of N G6-branes of M-theory on $T^6 \times \mathbb{R}$, in the large N limit. The low-energy limit of this is 6+1 dimensional SYM, but the theory must also contain extended objects that reduce to the strings of the $d = 5$ case in the decompactification limit. As we shall see, the theory contains membranes [15] which do precisely this. However, there is a difference between this and the $d = 5$ case: for $d = 5$ we obtain a string theory, but for $d = 6$ the membranes are not perturbative states, and so the theory is not a perturbative membrane theory. Thus the situation is similar to that of M-theory, which also contains membranes, but is not a perturbative membrane theory.

For $d \geq 7$, it is hard to generalise this picture due to the problems with having arbitrary numbers of Dd-branes. For $d = 8$, the D8-brane has been conjectured to arise from a 9-brane of M-theory [23]. This would suggest that the $8 + 1$ dimensional SYM arises from a 9+1 dimensional theory, and that the matrix theory is this 9+1 dimensional theory, that gives the strong-coupling limit of N D8-branes on $T^8 \times \mathbb{R}$. Further evidence in favour of this will be presented later. For $d = 7$, the D7-brane arises from a Kaluza-Klein monopole of M-theory on a T^2 that shrinks to zero size, where one circle is transverse and one longitudinal (or from the corresponding F-theory construction). If the M9-brane conjecture is correct, and if a D8-brane arises from its double dimensional reduction on a circle, then the D7-brane arises from an M9-brane on $T^7 \times T^2$ in the limit in which the T^2 shrinks to zero size. Understanding the matrix theory in this case would require a better understanding of these formulations of the D7-brane at finite coupling. The case $d = 9$ will be briefly discussed in section 5.

Toroidally compactified M-theory has a remarkable limit whose low energy effective description is in terms of $d + 1$ dimensional SYM. At least for $d \leq 5$, this limit gives a consistent interacting matter theory which is SYM for $d \leq 3$ and gives unexpected new theories for $d = 4, 5$. For $d \geq 6$, the limiting theory should still have an effective SYM description, but there are many unresolved issues, such as whether there is a limit in which gravity and the bulk fields decouple.

The purpose of this paper is to use the expected U-duality symmetries to try and learn something about the effective SYM theories that emerge in these limits of toroidally compactified M-theory, and then use the effective SYM theories to learn about matrix theory, assuming it exists for $d \geq 6$. It should be emphasized that the considerations in this paper are purely kinematical and concern the BPS states of the low energy effective braneworld-volume theories, and are independent of whether the matrix limits actually exist for $d \geq 6$. The usual analysis is for rectangular tori without background fields; U-duality requires general backgrounds with arbitrary torus metrics and background fields. The full matrix theory is expected to have 'manifest' U-duality symmetry. For $d \leq 3$, the matrix theory is just the $d+1$ dimensional SYM theory and the U-duality is indeed manifest. For $d > 3$, the $d+1$ dimensional SYM theory is only an effective description and the U-duality is not expected to be manifest in the SYM theory. Indeed, U-duality is manifest in the 6-dimensional self-dual tensor and string theories which are the matrix theories for $d = 4, 5$, but it is not a symmmetry of the $d+1$ dimensional SYM theory. Nevertheless, there is a remnant of U-duality in the SYM theory: it must couple to all the moduli that are related to the torus metric by U-duality. Thus U-duality gives important information about the terms that occur in the SYM effective action, and which terms in the D-brane action survive in the 'matrix theory limit'. This in turn gives information about the spectrum of BPS branes of the matrix theory, since these must arise as solitonic branes of the SYM theory; e.g. it implies the presence of $d-4$ branes, giving the 0-branes in 4+1 dimensions that correspond to Kaluza-Klein modes of the matrix theory, and the solitonic strings in 5+1 dimensions which correspond to the fundamental strings of the matrix theory.

For example, M-theory on T^3 should have $SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$ U-duality, implying that the corresponding $3+1$ dimensional SYM theory should have $SL(2,\mathbb{Z})$ symmetry. The appropriate limit of the D3-brane action includes a θ-angle coupling, and we learn that this is essential for the $SL(2,\mathbb{Z})$ S-duality, and that the theta-angle and the coupling constant take values in an $SL(2)/U(1)$ coset space. Of course, all this was already well-known for 4-dimensional SYM, but a similar analysis gives interesting predictions for higher values of d. In particular, we will learn that for $d = 6$, there is a generalised theta-angle (for certain 'instantonic 3-branes') that combines with the coupling constant to form a complex field taking values in an $SL(2)/U(1)$ coset space, and which is acted on by an $SL(2,\mathbb{Z})$ duality symmetry. In this way, useful information will be learned about the effective SYM description of these limiting theories, whenever the limits exist.

In each case, the SYM action that is found depends on the metric and certain constant anti-symmetric tensors on the torus. These can be thought of as generalised coupling constants and arise from the expectation values of background supergravity fields [25]. For example, for $d \geq 3$, the $D = d+1$ dimensional action includes the coupling to a $D-4$ form, whose Hodge dual is a 4-form X, and is of the form

$$\frac{1}{4g^2}\mathrm{tr}F_{mn}F^{mn} + \frac{1}{2}X^{mnpq}\mathrm{tr}F_{mn}F_{pq}\,. \tag{1.1}$$

Some of the consequences of such terms were considered in [25].

2. U-duality

M-theory compactified on T^d has U-duality group $E_d(\mathbb{Z})$ and scalars taking values in E_d/H_d, where H_d is the maximal compact subgroup of E_d. In each case, SYM on $\tilde{T}^d \times \mathbb{R}$ has a manifest $SL(d,\mathbb{Z})$ symmetry acting naturally on the torus, and the torus metric moduli lie in $\mathbb{R} \times SL(d)/SO(d)$. The SYM action includes the kinetic term, depending on the torus metric, and couplings to all the background fields related to this by E_d duality, so that it couples to all moduli in E_d/H_d. As the $SL(d)$ subgroup of E_d acts through torus diffeomorphisms, decomposing E_d/H_d into $SL(d)$ representations will tell us the tensor structure of the SYM coupling constants. We will consider each value of d in turn. For $d \leq 5$, the results of [1]–[12] are recovered, but the analysis makes interesting predictions for $d > 5$.

2.1. $d \leq 3, E_3 = SL(3) \times SL(2)$

The moduli space $SL(3)/SO(3) \times SL(2)/U(1)$ corresponds to metrics on T^3, plus one additional scalar, the θ-angle C_0, which combines with the torus volume (which determines the SYM coupling constant g) to take values in $SL(2)/U(1)$. The lagrangian includes the terms

$$L \sim \frac{1}{g^2}\mathrm{tr}F \wedge *F + C_0 \mathrm{tr}F \wedge F \tag{2.1}$$

and the geometric $SL(3,\mathbb{Z})$ acting on the torus combines with the $SL(2,\mathbb{Z})$ S-duality, so that the full $E_3(\mathbb{Z})$ U-duality is manifest. The U-duality is also manifest for $d < 3$, as follows immediately by taking suitable limits.

2.2. $d = 4, E_4 = SL(5)$

The parameter space of the theory is $SL(5)/SO(5)$, while the moduli of the 4-torus are in $\mathbb{R} \times SL(4)/SO(4)$; the extra parameters in $SL(5)/SO(5)$ are in a **4** of $SL(4)$ and so are associated with a 1-form C_1 on $\tilde{T}^4$. (A 3-form coupling through $C_3 \wedge F$ would have given the same counting, but it will be argued in section 3 that it is the 1-form that gives the correct SYM theory, as used in [9]–[12].) The lagrangian for $4+1$ SYM then includes the terms

$$L \sim \frac{1}{g^2}\mathrm{tr}F \wedge *F + C_1 \wedge \mathrm{tr}F \wedge F\,. \tag{2.2}$$

However, *any* $SL(4)\times\mathbb{R}$ subgroup of $SL(5)$ can be associated with a 4-torus in this way, so that different 4-tori emerge from different limits of the moduli space. This suggests that the full theory be formulated on a 5-torus, and be given by a 5+1 dimensional field theory on $\tilde{T}^5 \times \mathbb{R}$. It was argued in [9]–[12] that this is indeed the case, and that the matrix theory is given by the (2,0) supersymmetric self-dual tensor theory on $\tilde{T}^5 \times \mathbb{R}$. This theory has an ultra-violet fixed point [12], and the scale invariance then implies that the theory is independent of the volume of the $\tilde{T}^5$, explaining the absence of an extra $\mathbb{R}$ factor in the moduli space $SL(5)/SO(5)$. The self-duality of the tensor theory implies that it has no adjustable coupling constant. The expected 4+1 SYM emerges in the appropriate limits [9]–[12].

2.3. $d = 5, E_5 = SO(5,5)$

The parameter space of the theory is $SO(5,5)/SO(5)\times SO(5)$, while the moduli of the 5-torus are in $\mathbb{R}\times SL(5)/SO(5)$; the extra parameters in $SO(5,5)/SO(5)\times SO(5)$ are in a **10** of $SL(5)$ and so are associated with a 2-form C_2 on $\tilde{T}^5$. (A 3-form on $\tilde{T}^5$ would also be in the **10**, but the D5-brane does not couple to a 5-form.) The lagrangian for $5+1$ SYM then includes the terms

$$L \sim \frac{1}{g^2}\mathrm{tr}F\wedge *F + C_2\wedge \mathrm{tr}F\wedge F\,. \tag{2.3}$$

This is part of the low-energy effective action of the (1,1) supersymmetric string theory in 5+1 dimensions proposed in [12]. The strings coupling to C_2 arise as solitons of the low-energy effective theory, which in this case correspond to YM instantons. It was proposed in [10],[12] that M-theory on T^5 corresponds to this string theory on $\tilde{T}^5\times\mathbb{R}$. The matrix theory limit requires going to strong string coupling, so that it is useful to first perform an $SL(2,\mathbb{Z})$ duality transformation which takes the D5-brane to the NS 5-brane so that the limit is now one of weak string coupling. The theory emerging in this limit is a 6-dimensional (1,1) supersymmetric non-critical string theory with the SYM theory as its zero slope limit. For the D5-brane, C_2 is the RR 2-form while for the NS 5-brane it is the NS 2-form. This theory is related by T-duality to the (2,0) string on a 5-torus [12] whose low-energy limit is the tensor theory that arose for $d=4$, so that either string theory on a 5-torus can be used to describe M-theory on T^5.

2.4. $d = 6, E_6$

This is the first case in which we learn something new. The parameter space of the theory is $E_6/USp(8)$, while the moduli of the 6-torus are in $\mathbb{R}\times SL(6)/SO(6)$. E_6 has a maximal subgroup $SL(6)\times SL(2)$, under which $\mathbf{78}\to(\mathbf{1},\mathbf{3})+(\mathbf{35},\mathbf{1})+(\mathbf{20},\mathbf{2})$. The parameter space $E_6/USp(8)$ then decomposes into $SL(6)/SO(6)\times SL(2)/U(1)$, plus an extra 3-form C_3 on $\tilde{T}^6$ in the **20** of $SL(6)$. The 3-form couples through a C_3F^2 term, but the coupling to the extra scalar χ that combines with the coupling constant g (or, equivalently, the torus volume) to parameterise the coset space $SL(2)/U(1)$ is more problematic. It will be argued in section 3 that this coupling is through a term $(DX)^3F^2$, where the X are the three adjoint scalars in the $6+1$ dimensional SYM multiplet, and $DX = dX + [A,X]$. The resulting lagrangian for $6+1$ SYM then includes the terms

$$L \sim \frac{1}{g^2}\mathrm{tr}F\wedge *F + C_3\wedge \mathrm{tr}(F\wedge F) + \chi\mathrm{Str}\,(DX\wedge DX\wedge DX)\wedge F\wedge F\,, \tag{2.4}$$

where Str denotes the symmetrised trace. The term $\chi(DX)^3F^2$ involves an 'axionic' scalar χ and is obtained from 10 dimensions by the reduction of the term $\chi\mathrm{tr}F^5$, so that

χ is a generalised θ angle. Such a $\chi \mathrm{tr} F^5$ term arises formally as part of the D9-brane action.

In this theory, YM instantons give solitons of the low-energy theory that correspond to membranes coupling to C_3. Thus the 6+1 dimensional theory contains membranes and gives rise to the strings of the (1,1) theory in 5+1 dimensions on double dimensional reduction, and to membranes which are expected to be Dirichlet branes of the non-critical string theory on simple dimensional reduction. The 'θ-angle' χ couples to certain instantons which will be discussed in section 4.

2.5. $d = 7, E_7$

The parameter space of the theory is $E_7/SU(8)$, while the moduli of the expected 7-torus are in $\mathbb{R} \times SL(7)/SO(7)$. The 70 parameters in $E_7/SU(8)$ then decompose into the **27+1** moduli of T^7 in $\mathbb{R} \times SL(7)/SO(7)$, together with a **35** and a **7** of SL(7). The **35** could be a 4-form or a 3-form on T^7, while the **7** could be a vector or a 6-form. The D7-brane does not couple to a vector or 3-form, but with a 4-form and a 6-form, one can write the following action for $7+1$ SYM

$$S = \int \frac{1}{g^2} \mathrm{tr} F \wedge *F + C_4 \wedge \mathrm{tr} F \wedge F + C_6 \wedge \mathrm{tr} F \,. \tag{2.5}$$

Other possibilities for the action will be discussed in section 3. For the action (2.5), the YM instantons correspond to 3-branes coupling to C_4.

In fact, E_7 contains $SL(8)$ as a maximal subgroup and under this the parameter space decomposes into $SL(8)/SO(8)$ together with a 35^+ of $SL(8)$, corresponding to a self-dual 4-form. The situation is then similar to that for $d = 4$. Any $SL(7)$ subgroup of $SL(8)$ is associated with a 7-torus, suggesting that the theory could be an 8+1 dimensional theory on $T^8 \times \mathbb{R}$. Moreover, this theory should be scale invariant, to account for the absence of a modulus for the volume of the 8-torus. The 8+1 dimensional theory should contain 3-branes coupling to the self-dual 4-form on T^8. However, it is problematic to obtain a self-dual 4-form on T^8 from a covariant field on $8+1$ dimensions. Moreover, for a standard Kaluza-Klein picture to apply, one would expect the **7** to correspond to a vector, not a 6-form, with the vector interpreted as a gravi-photon coupling to 0-branes that become light in the decompactification limit.

2.6. $d = 8, E_8$

Finally, in this case the parameter space is $E_8/SO(16)$, while the moduli of the expected 8-torus are in $\mathbb{R} \times SL(8)/SO(8)$. The 128 moduli in $E_8/SO(16)$ correspond to the **35+1** moduli of $\tilde{T}^8$, plus parameters in the **56,28,8** representations of $SL(8)$. The **56** could correspond to a 5-form or 3-form on $\tilde{T}^8$, with the coupling $C_5 \wedge F^2$ or $C_3 \wedge F^3$, while the **28** could correspond to a 2-form or 6-form, with a coupling $C_2 \wedge DX \wedge F^3$ or $C_6 \wedge DX \wedge F$. The **8** could correspond to a 1-form or 7-form on $\tilde{T}^8$, with the coupling

$A_1 \wedge *J$ or $C_7 \wedge F$, where J is the conserved current $J_m = F_{mn} D^n X$. If a 5-form, 2-form and 7-form are chosen, for example, an action of the form

$$S = \int \mathrm{Str}\left[\frac{1}{g^2} F \wedge *F + C_5 \wedge F \wedge F + C_7 \wedge F + C_2 DXF^3\right] \tag{2.6}$$

can be written down; another possibility with background 1, 5 and 6 forms is

$$S = \int \mathrm{Str}\left[\frac{1}{g^2} F \wedge *F + C_5 \wedge F \wedge F + \omega_6 \wedge DX \wedge F + A_1 \wedge DX \wedge *F\right] \tag{2.7}$$

and other possibilites will be considered in section 3.

However, E_8 in fact contains $SL(9)$ as a maximal subgroup. As before, any $SL(8)$ subgroup of $SL(9)$ can be associated with an 8-torus, suggesting that the theory could be a scale-invariant 9+1 dimensional theory on $T^9 \times \mathbb{R}$, in which case the moduli correspond to the **44** constant volume metrics on T^9 in $SL(9)/SO(9)$, together with a 6-form C_6 in the **84** of $SL(9)$. A 9+1 dimensional SYM lagrangian can be written with a coupling to a metric and 6-form, containing the following terms:

$$L \sim \frac{1}{g^2} \mathrm{tr}(F \wedge *F) + C_6 \wedge \mathrm{tr}(F \wedge F)\,. \tag{2.8}$$

The supersymmetric lagrangian is then that of $D = 10$ SYM, plus a topological term $C_6 \wedge F^2$. Remarkably, the reduction of this to 8+1 dimensions gives the candidate lagrangian (2.7) (at least to linearised order). Indeed the charge J_0 with respect to the background Maxwell field A_1 is just the momentum in the compactified direction. Thus if the action is (2.7), there is the possibility that 0-branes of the 8+1 dimensional theory carrying the charge J_0 could be reinterpreted as Kaluza-Klein modes for a compactified 10-dimensional theory.

2.7. Summary

For each d, the low-energy effective action is SYM on $\tilde{T}^d \times \mathbb{R}$. The coupling constants consist of (i) the torus metric and SYM coupling constant (which can be absorbed into the volume of the torus), taking values in $\mathbb{R} \times SL(d)/SO(d)$ (ii) a $d-3$ form for $d \geq 3$ (or its dual, a 4-form) (iii) a $d-6$ form for $d \geq 6$ (or its dual, a 7-form) and (iv) a 1-form for $d = 8$. Thus for $d < 8$ we obtain $d+1$ dimensional SYM coupling to a metric, a 4-form and a 7-form. The $d-3$-form couples through a term of the form

$$C_{D-4} \wedge F \wedge F\,, \tag{2.9}$$

which can be written in terms of the 4-form $X = *C_{D-4}$ as in (1.1). Instantons in 4 Euclidean dimensions give rise to $d-4$ brane solitons of the SYM theory that carry charge with respect to the potential C_{D-4}. For $D = 6$, the 1-brane solitons of the SYM theory are identified with the fundamental strings of the 6-dimensional string theory and for $D = 5$, the 0-branes are Kaluza-Klein modes of the 6-dimensional tensor theory. For $d = 7$, there is either an extra vector which would couple to 0-branes, or an extra 6-form which would couple to 5-branes. For $d = 8$ there is a vector or 7-form, coupling to 0-branes or 6-branes, and a 2-form or 6-form coupling to strings or 5-branes. Furthermore, U-duality suggests that for $d = 4, 7, 8$ the full theory could in fact be a scale-invariant theory in $d + 2$ dimensions, and these are precisely the cases in which there may be 0-brane solitons which could represent Kaluza-Klein modes. It is clearly important to find further restrictions on the cases where the group theory does not give a unique prediction, and this will be done in the next section.

3. D-brane actions and U-duality

The matrix model for M-theory on T^d or a Dd-brane wrapped on T^d is a SYM theory coupling to the metric on T^d, so that the metric provides coupling constants of the theory, taking values in $GL(d)/SO(d)$. If the U-duality group E_d is to be realised on the theory, then the SYM should couple to all the scalars related to the metric coupling constants by U-duality, so that the SYM should couple to scalars in E_d/H_d where H_d is the maximal compact subgroup of E_d. Moreover, the conjecture of [3] suggests that in fact the SYM theory at finite N realises the U-duality group E_{d+1}. The D-brane couples to all scalars in E_{d+1}/H_{d+1}, and the matrix theory of [3] is obtained as a limit of this D-brane theory [16],[17]. We will now investigate these couplings.

The D-brane action for a single p-brane in a supergravity background with constant fields is a $D = p + 1$ dimensional action including the terms [26], [27], [28]

$$S = \int d^D x \, e^{-\Phi} \sqrt{\det\left(g_{mn} + \mathcal{F}_{mn}\right)} + \int \left[C_D + C_{D-2} \wedge \mathcal{F} + C_{D-4} \wedge \mathcal{F} \wedge \mathcal{F} + \right. \\ \left. + C_{D-6} \wedge \mathcal{F} \wedge \mathcal{F} \wedge \mathcal{F} + ... + C_{D-2r} \wedge \mathcal{F}^r\right] , \tag{3.1}$$

where $\mathcal{F}_{mn} = F_{mn} - b_{mn}$, r is the integer part of $D/2$, g_{mn}, b_{mn} are the pull-backs of the metric G_{MN} and the NS-NS 2-form gauge field B_{MN}

$$g_{mn} = G_{MN} \partial_m X^M \partial_n X^N, \qquad b_{mn} = B_{MN} \partial_m X^M \partial_n X^N \tag{3.2}$$

and the C_r are r-forms arising from the background expectation values of the pull-backs of RR gauge fields, so that

$$C_r = C_{M_1 ....M_r} \partial_{m_1} X^{M_1} ... \partial_{m_r} X^{M_r} d\sigma^{m_1} \wedge ... \wedge d\sigma^{m_r} , \tag{3.3}$$

where $M, N = 0, 1, ...9$ are space-time indices and $m, n = 0, 1, ..., p$ are p-brane world-volume indices. For $r > 4$, the C_r are the dual potentials; the field equation for C_r with $r \leq 4$ is of the form $dG = 0$, where $G = *dC_r + ...$ is the $9 - r$ form field strength, and the dual potential C_{8-r} is the potential for G, $G = dC_{8-r}$ [26]. Similar actions arise in matrix theory. From the point of view of the SYM theory, the forms C_r are coupling constants. On going to static gauge, the X^M are split into coordinates X^m, which are identified with the σ^m so that $\partial_m X^n = \delta_m{}^n$, together with the transverse coordinates X^α where $\alpha = 1, ..., 9-p$. Then the tensor $C_{M_1 ... M_r}$ splits into a set of forms $C_{m_1 ... m_t \alpha_1 ... \alpha_{r-t}}$ for various t, and C_r (3.3) becomes

$$
\begin{aligned}
C_r &= \sum_t C_{m_1 ... m_t \alpha_1 ... \alpha_{r-t}} \partial_{m_{t+1}} X^{\alpha_1} ... \partial_{m_r} X^{\alpha_{r-t}} d\sigma^{m_1} \wedge ... \wedge d\sigma^{m_r} \\
&= \sum_t (C_t)_{\alpha_1 ... \alpha_{r-t}} dX^{\alpha_1} \wedge dX^{\alpha_2} \wedge ... \wedge dX^{\alpha_{r-t}} \,,
\end{aligned}
\tag{3.4}
$$

where the t-form C_t is defined by

$$
(C_t)_{\alpha_1 ... \alpha_{r-t}} = C_{m_1 ... m_t \alpha_1 ... \alpha_{r-t}} d\sigma^{m_1} \wedge ... \wedge d\sigma^{m_t} \,. \tag{3.5}
$$

Thus (3.1) contains terms such as

$$
(C_t)_{\alpha_1 ... \alpha_s} dX^{\alpha_1} \wedge dX^{\alpha_2} \wedge ... \wedge dX^{\alpha_s} \wedge \mathcal{F}^m \tag{3.6}
$$

with $D = t + s + 2m$. Similarly, g_{mn}, b_{mn} become

$$
\begin{aligned}
g_{mn} &= G_{mn} + 2G_{\alpha(m} \partial_{n)} X^\alpha + G_{\alpha\nu} \partial_m X^\alpha \partial_n X^\nu , \\
b_{mn} &= B_{mn} + 2B_{\alpha(m} \partial_{n)} X^\alpha + B_{\alpha\nu} \partial_m X^\alpha \partial_n X^\nu \,.
\end{aligned}
\tag{3.7}
$$

For N D-branes, the gauge symmetry becomes $U(N)$ and the gauge fields and the transverse scalars X^α take values in the adjoint representation of $U(N)$. The derivative $dX^\alpha = \partial_m X^\alpha d\sigma^m$ is replaced by the covariant derivative $DX^\alpha = D_m X^\alpha d\sigma^m$, where

$$
D_m X^\alpha = \partial_m X^\alpha + [A_m, X^\alpha] \,. \tag{3.8}
$$

Terms in (3.1) that are independent of X become of the form $C_s \mathrm{tr} \mathcal{F}^t$ where now

$$
\mathcal{F}_{mn} = F_{mn} - b_{mn} \mathbb{1} \tag{3.9}
$$

but (3.6) has a number of possible non-abelian generalisations. The simplest guess would be

$$
(C_t)_{\alpha_1 ... \alpha_s} \mathrm{Str} \left[DX^{\alpha_1} \wedge DX^{\alpha_2} \wedge ... \wedge dX^{\alpha_s} \wedge \mathcal{F}^m \right] \tag{3.10}
$$

and this is supported by [29].

The D-brane action (3.1) defines the coupling of the D-brane to all of the 128 bosonic degrees of freedom of the type II string, consisting of G_{MN}, B_{MN}, Φ together with the RR gauge fields; these can be thought of as the generalised coupling constants of the SYM theory. For compactification of the type II theory on T^d to $\mathbb{R}^{9-d,1}$, the 128 bosonic degrees of freedom can be split into fields on $\mathbb{R}^{9-d,1}$ of various spins. In particular, the scalars are the moduli for type II on T^d, corresponding to the metric moduli plus the moduli of flat anti-symmetric tensor gauge fields on T^d. These parameterise the coset space E_{d+1}/H_{d+1} where H_{d+1} is the maximal compact subgroup of E_{d+1}.

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In [3] it was conjectured that M-theory on a null circle of finite radius R_{11} in the discrete light-cone gauge is described by a matrix model with $U(N)$ gauge symmetry for finite N. In [16], [17], this was related to a limit of the theory of N D-branes, so that M-theory on $S^1 \times T^d$ (with null S^1) should be described by $U(N)$ gauge theory (for finite N) on $\tilde{T}^d \times \mathbb{R}$. The Dd-brane theory has $E_{d+1}(\mathbb{Z})$ symmetry instead of the $E_d(\mathbb{Z})$ symmetry expected from the old matrix model conjecture and it should couple to fields in the coset space E_{d+1}/H_{d+1} (at least before taking the limits of [16], [17]), not just the coset E_d/H_d. The coupling to these fields is precisely what is found in the D-brane action. The matrix model action for infinite R_{11} is obtained by a truncation to the terms coupling to a E_d/H_d subspace of E_{d+1}/H_{d+1}. In particular, this truncation involves dropping the dilaton coupling, as the matrix model limit involves the weak string coupling limit $\tilde{g} \to 0$. In the following, we will investigate which fields should be dropped in this limit to obtain a SYM theory in $d+1$ dimensions coupling to all moduli in E_d/H_d. Although the truncations can be found by careful consideration of the matrix theory limit, it is of interest to see how much can be derived using kinematics and group theory, as these methods can be applied more generally, including cases in which the matrix theory limit is not yet understood (and may not exist, at least in the usual form).

Our conventions are that X^M are the space-time coordinates in $\mathcal{D}=10$ or $\mathcal{D}=11$ dimensions (depending on context) and that on compactification these are split into $X^M = (X^\mu, X^i)$, where X^i ($i=1,...,d$) are coordinates on T^d, and X^μ are space-time coordinates ($\mu = 0,1,2,...,\mathcal{D}-d$). On a d-brane, the coordinates can instead be split into $X^M = (X^m, X^\alpha)$ where X^m are longitudinal coordinates ($m=0,1,...,d$) and X^α are transverse ($\alpha = 1,...,\mathcal{D}-d-1$), so that for a Dd-brane wrapped on T^d, $X^m = (X^0, X^i)$ and $X^\mu = (X^0, X^\alpha)$. Note that each field that emerges from an r-form gauge field C_r on toroidal reduction can also be regarded as coming from its dual, $\tilde{C}_{8-r}$. For example, for the type IIB theory reduced on T^d, there are $d(d-1)/2$ scalars C_{ij} that arise from a two-form C_2 on T^d. In $10-d$ dimensions, an $8-d$ form is dual to a scalar, and the $d(d-1)/2$ scalar degrees of freedom could instead be thought of as arising from the dual potential C_6, giving the $d(d-1)/2$ $8-d$ forms $C_{\mu_1 \dots \mu_{8-d} i_1 \dots i_{d-2}}$. It is important not to double-count such dual realisations of the same degrees of freedom in the following. When there is an ambiguity, we will choose the dual forms that have local couplings to the Yang-Mills theory when it is weakly coupled.

3.1. $d=3$

Consider first the case $d=3$. The D3-brane wrapped on T^3 has an action which couples to the constant fields $G_{ij}, B_{ij}, C_{ij}, C_0, \Phi$ on the torus, transforming in the **5+1+3+3+1+1** of $SL(3)$ giving 14 scalars, which parameterise the coset space $E_4/H_4 = SL(5)/SO(5)$. We wish to find the restriction to scalars in the **5+1+1** of $SL(3)$ parameterising $SL(3)/SO(3) \times SL(2)/U(1)$ to obtain the matrix theory for

M-theory on T^3. The **5** clearly corresponds to the metric, and the weak coupling limit requires that the dilaton be dropped, so that the **1+1** must correspond to the size of the torus and the axion C_0. The SYM coupling constant is proportional to the volume of the 3-torus, and we learn that g and C_0 together parameterise the coset space $SL(2)/U(1)$ and that there is an $SL(2,\mathbb{Z})$ duality symmetry which acts on them. The action includes the terms (2.1). Thus the matrix theory conjecture together with the conjectured U-duality of the 8-dimensional string theory [21] predicts the S-duality of 4-dimensional SYM. Although this S-duality is not a surprise here, this will have interesting generalisations for higher d.

3.2. $d = 4$

For $d = 4$, the D4-brane couples to $G_{ij}, B_{ij}, C_{ijk}, C_i, \Phi$ in the **9+1+6+4+4+1** of $SL(4)$ giving 25 scalars, which parameterise the coset space $E_5/H_5 = SO(5,5)/SO(5) \times SO(5)$. To obtain the restriction to $SL(5)/SO(5)$, we need to truncate to 14 scalars in the **9+1+4** representation of $SL(4)$. This should not include the dilaton, so that the only choice is between C_i and C_{ijk} to give the **4**. The D4-brane couples to both of these, through C_1F^2 and C_3F. For the $d = 3$ case, we saw that it was necessary to keep C_0 and throw out C_{ij}, so to recover this case in the appropriate decompactification limit requires keeping C_i and not C_{ijk} in this case. We are thus led to the the action given by (2.2), with the coupling constant determined by the torus volume.

3.3. $d = 5$

For $d = 5$, the D5-brane couples to $G_{ij}, B_{ij}, C_{ij}, C_{ijkl}, C_0, \Phi$ in the **14+1+10+10+5+1+1** of $SL(5)$ giving 42 scalars, which parameterise the coset space $E_6/H_6 = E_6/USp(8)$. To obtain the restriction to $SO(5,5)/SO(5) \times SO(5)$, we need to truncate to 25 scalars in the **14+1+10** representation of $SL(5)$. This should not include the dilaton, and the volume of the metric should be chosen instead of C_0 to give the extra singlet since the torus volume was included for $d < 5$. The **10** could come from either the NS two-form B_{ij} or the RR two-form C_{ij}. The natural choice is to exclude the coupling to B_{ij} because the $d = 3, 4$ cases did not include it, and these cases should be recovered in the limit in which T^5 decompactifies to $T^3 \times \mathbb{R}^2$ or $T^4 \times \mathbb{R}$. Then restricting to G_{ij}, C_{ij} gives the coupling to 25 scalars in the **14+1+10** representation of $SL(5)$ parameterising $SO(5,5)/SO(5) \times SO(5)$, with the action given by (2.3). This is the action for the D5-brane; a type IIB $SL(2,\mathbb{Z})$ transformation takes this to the NS 5-brane and replaces C_{ij} with B_{ij}, to give a coupling $B \wedge F \wedge F$, as in [10], [12], and it is in this form of the theory that the appropriate matrix theory limit is that of weak string coupling [10], [12].

3.4. $d = 6$

For $d = 6$, the 6-brane couples to $G_{ij}, B_{ij}, C_{ijk}, C_i, \Phi$ in the **20+1+15+20+6+1** of $SL(6)$ giving 63 scalars, instead of the 70 that would be needed for the coset space $E_7/SU(8)$. The remaining scalars arise from the dualisation of anti-symmetric tensor gauge fields. The bosonic spectrum of type IIA compactified on T^6 contains, in addition to the 63 scalars, **1+6** 2-forms from $B_{\mu\nu}$ and $C_{\mu\nu i}$ and one 3-form $C_{\mu\nu\rho}$, together with a metric $G_{\mu\nu}$ and **6+6+15+1** =28 vectors from $G_{\mu j}, B_{\mu j}, C_{\mu jk}, C_\mu$. For IIA on T^6, the dimensionally reduced theory is 4-dimensional and in 4 dimensions 2-forms are dual to scalars, so that the 63 scalars together with the extra 7 scalars arising from dualising the 2-forms parameterise the 70-dimensional coset space $E_7/SU(8)$. Equivalently, these extra **1+6** scalars can be viewed as coming from the coupling to the components C_{ijklm} of the potential C_5 (dual to the 3-form potential C_3) and to B_{ijklmn} (dual to the NS 2-form). Any of the scalars arising from an r-form gauge field can equivalently arise as a 2-form obtained from the dual $8 - r$ form potential. For example, the **6** scalars C_i could be represented by the **6** dual 2-forms $C_{\mu\nu ijklm}$ obtained from C_7, the dual to C_1. There are similar dual forms of the compactifications for other d.

Restricting to G_{ij}, C_{ijk} gives the coupling to **20+1+20** =41 scalars. For M-theory on T^6, the compactified theory is 5-dimensional with 42 scalars in $E_6/USP(8)$ in the **20+1+20+1** of $SL(6)$, so that an extra singlet is needed. There are no other singlet scalars to choose from, and the 2-forms do not give scalars in 5-dimensions. However, in 5 dimensions 3-form gauge fields are dual to scalars. Dualising the single 3-form $C_{\mu\nu\rho}$ gives an extra scalar, to give the 42 scalars parameterising $E_6/USp(8)$. This fits in with the M-theory picture, where on compactification of M-theory on T^6, the **20+1+20+1** scalars come precisely from $G_{ij}, A_{ijk}, A_{\mu\nu\rho}$ where here A_{MNP} is the 3-form of 11-dimensional supergravity. Thus both M-theory and matrix theory give 41 scalars (taking values in G/H where $G = GL(6,\mathbb{R})/SO(6) \times \mathbb{R}^{20}$ [30]) and a 3-form. (In fact, the D6-brane also couples to 3-forms $C_{\mu\nu\rho ij}, C_{\mu\nu\rho ijkl}$ arising from C_5, C_7, but these are both in the **15** of $SL(6)$ and so do not give singlets.) Thus we learn that the matrix model for M-theory on T^6 in the limit $R_{11} \to \infty$ has as its low-energy effective theory SYM coupled to the scalars G_{ij}, C_{ijk} and the 3-form $C_{\mu\nu\rho}$. The coupling to $C_{\mu\nu\rho}$ follows from the D-brane action (3.1) (with abelian gauge group), and is of the form

$$C_{MNR}(dX^M \wedge dX^N \wedge dX^R \wedge F \wedge F)\,. \tag{3.11}$$

The 10 X^M split into three transverse X's, X^α ($\alpha, \beta = 1, 2, 3$) and the 6+1 longitudinal X's, X^m, which can be identified with the world-volume coordinates on going to static gauge, so that the action becomes

$$C_{\alpha\beta\gamma}(dX^\alpha \wedge dX^\beta \wedge dX^\gamma \wedge F \wedge F)\,. \tag{3.12}$$

Now $C_{\alpha\beta\gamma} = \chi\epsilon_{\alpha\beta\gamma}$ for some χ, and the SYM coupling becomes

$$\chi\epsilon_{\alpha\beta\gamma}(dX^\alpha \wedge dX^\beta \wedge dX^\gamma \wedge F \wedge F)\,. \tag{3.13}$$

The natural candidate for the non-abelian generalisation of this is [29]

$$\chi\epsilon_{\mu\nu\rho}\mathrm{Str}(DX^\mu DX^\nu DX^\rho F^2)\,. \tag{3.14}$$

This gives the action (2.4).

3.5. $d = 7$

For $d = 7$, the D7-brane couples to $G_{ij}, B_{ij}, C_{ij}, C_{ijkl}, C_0, \Phi$ in the **27+1+21+21+35+1+1** of $SL(7)$ giving 107 scalars. In addition there are **7+7+7** =21 vectors $G_{i\mu}, B_{i\mu}, C_{i\mu}$. In the 3-dimensional theory obtained by compactifying the type IIB theory on T^7, vectors are dual to scalars and dualising these 21 vectors give 21 additional scalars and a total of $107 + 21 = 128$ scalars parameterising $E_8/SO(16)$. The **7+7** vectors $B_{i\mu}, C_{i\mu}$ could instead be given as scalars arising from the components B_{ijklmn}, C_{ijklmn} of the dual potentials B_6, C_6. M-theory on T^7 should couple to 70 scalars in the coset space $E_7/SU(8)$ and as we saw in section 2, the group theory implies that these should be in the **27+1+35+7** of $SL(7)$. The scalars in the **27+1+35** representation should come from G_{ij}, C_{ijkl} but there is more than one possibility for the **7**. The simplest is that these should arise from the scalars C_{ijklmn}, through the coupling $C_6 \wedge F$. However, in 4 dimensions, 2-forms are dual to scalars and the missing degrees of freedom could arise from **7** 2-forms. The potentials $B_2, C_2, C_4, C_6, B_6, C_8$ give rise to **1+1+21+35+35+7** 2-forms (e.g C_4 gives the 21 2-forms $C_{ij\mu\nu}$). These do not give rise to propagating degrees of freedom in 3 dimensions but are dual to scalars in 4 dimensions. Thus the only possibility is through the coupling to the components $C_{ijklmn\mu\nu}$ of C_8, which couples to the 2 transverse scalars X^α through the coupling $\omega_6 \wedge DX \wedge DX$ where $C_{ijklmn\alpha\beta} = \omega_{ijklmn}\epsilon_{\alpha\beta}$. The fact that M-theory on T^7 gives 63 scalars from constant metric G_{ij} and 3-form gauge fields A_{ijk} on T^7 and 7 space-time 2-form gauge fields $A_{i\mu\nu}$ might be construed as evidence in favour of the coupling to C_8, which gives the coupling to 7 2-forms. In both cases, there is a 6-form which couples to a 5-brane inside the 7-brane, as will be discussed in section 4.

3.6. $d = 8$

For $d = 8$, the D8-brane couples to the background fields G_{ij}, B_{ij}, C_{ijk}, C_i, Φ, which transform in the **35+1+28+56+8+1** representations of $SL(8)$, giving 129 moduli. Alternatively, the **56+8** could be thought of as arising form the dual potentials $C_{ijklm}, C_{ijklmnp}$. In three dimensions, vectors are dual to scalars and there are vectors in the **8+8+1+28+70+28+1+56** of $SL(8)$ from $G_{\mu j}, B_{\mu j}, C_\mu, C_{\mu jk}, C_{\mu jklm}, C_{\mu jklmnp}$, $C_{\mu ijklmnpq}, B_{\mu ijklm}$. M-theory on T^8 gives a 3-dimensional theory with scalars in the coset $E_8/SO(16)$, which has 128 scalars which are in the **35+1+28+56+8** of $SL(8)$. The **35+1** should arise from G_{ij}, while the **56** could arise from C_{ijk}, C_{ijklm} or $B_{\mu ijklm}$. The fact that all cases with $d < 8$ included the coupling $C_{d-3} \wedge F \wedge F$ suggests that the coupling to C_{ijklm} is required here. The **28** could arise from B_{ij} or $C_{\mu jk}$ or $C_{\mu jklmnp}$;

however, the coupling to B_{ij} can be excluded because it didn't occur for $d < 8$. Here there is one transverse scalar X so in static gauge α takes only one value, so writing $\omega_{jk} = C_{\alpha jk}$, $\omega_{jklmnp} = C_{\alpha jklmnp}$, the relevant couplings are $\omega_2 DXF^3$ or $\omega_6 DXF$. Finally, the **8** could arise from $C_{ijklmnp}$, $G_{\mu j}$ or $B_{\mu j}$. In the first case, the coupling would be $C_7 \wedge F$, while for the remaining two the linearised form of the coupling to the vector A_i would be of the form $A \wedge DX \wedge *F$. The fact that M-theory compactified on T^8 gives **35+1+56** scalars and **28+8** vectors (before duality transformations) suggests the coupling to the **35+1+56** scalars G_{ij}, C_{ijklm} together with the **28** vectors ω_2 or ω_6 and the **8** vectors $G_{\mu j}$ or $B_{\mu j}$. There are four choices of such action, one of which has the linearised form

$$S = \int \frac{1}{g^2} F \wedge *F + C_5 \wedge F \wedge F + \omega_6 \wedge DX \wedge F + A_1 \wedge DX \wedge *F \tag{3.15}$$

(another is the action obtained from this by replacing $\omega_6 DXF$ with $\omega_2 DXF^3$.) It is intriguing that the action (3.15) could be obtained by reducing the 10-dimensional action

$$S = \int \frac{1}{g^2} F \wedge *F + C_6 \wedge F \wedge F \tag{3.16}$$

on S^1, so that the 9-dimensional theory on T^8 arises from the 10-dimensional theory on T^9, and the $SL(9)$ subgroup of E_8 arises geometrically. The YM instantons give rise to 5-branes coupling to C_6, which reduce to give 4- and 5-branes in 8+1 dimensions.

4. New instantons and branes

In $6+1$-dimensional SYM, we have been led to consider the coupling

$$\int \chi \, \mathrm{Str} DX^3 \wedge F^2 \,. \tag{4.1}$$

This has many similarities with the term θF^2 in $3+1$ dimensions. First, we shall consider solutions of the Euclidean version of 7-dimensional SYM, whose bosonic sector consists of 3 adjoint-valued scalars X^α ($\alpha = 1, 2, 3$) coupled to Yang-Mills, with the usual potential $\mathrm{tr}([X^\alpha, X^\beta])^2$. Consider the following solution in $\mathbb{R}^7$. Splitting the coordinates x^m into coordinates x^α on $\mathbb{R}^3$ and x^i on $\mathbb{R}^4$, we choose $A_\alpha = 0$ and A_i the connection for an instanton solution on $\mathbb{R}^4$, and choose $X^\alpha = x^\alpha t^\alpha$ with no sum over α, where t^α are any 3 mutually commuting Lie algebra generators (not necessarily distinct) with product $t^1 t^2 t^3 = T$. Then the coupling (4.1) becomes

$$\int \chi \eta \wedge \mathrm{tr}(TF \wedge F) \,, \tag{4.2}$$

where $\eta = \frac{1}{6} \epsilon_{\alpha\beta\gamma} dx^\alpha \wedge dx^\beta \wedge dx^\gamma$ is the volume form on $\mathbb{R}^3$. For example, taking a gauge group $G = U(N)$, we can take an $SU(N)$ instanton on $\mathbb{R}^4$ and take $t^1 = t^2 = t^3 = T$ all as the $U(1)$ generator, so that 4.2 reduces to $\int \chi \eta \wedge \mathrm{tr} F^2$ and, for constant χ, is proportional to the second Chern class of the instanton. Then the coupling constant χ is a θ-angle term corresponding to these Euclidean solutions. Note that we could continue this solution back to a Minkowski space solution with one of the scalars X^α depending linearly on time. This solution can be thought of as a brane with 3 Euclidean dimensions, located at a point in $\mathbb{R}^4$.

The term (4.1) comes from the dimensional reduction of the term

$$\int \chi \mathrm{tr} F^5 \tag{4.3}$$

in ten dimensions. Any YM configuration on $T^3 \times M_7$ (where M_7 is a 7-manifold) with non-zero 5-th Chern class, $\int \mathrm{tr} F^5 \neq 0$, and which is independent of the toroidal directions can be reduced to a configuration for which the term (4.1) is non-zero. For example, the configuration considered above on $\mathbb{R}^4 \times T^3$ (with the x^α directions compactified) arises from the ten-dimensional configuration on $\mathbb{R}^4 \times T^6$ with an $SU(N)$ instanton on $\mathbb{R}^4$ and a magnetic monopole on each of the three T^2 i.e. on each T^2 there is a configuration with non-zero first Chern class, $\int \mathrm{tr} F \neq 0$.

The dimensional reduction of (4.3) gives the term

$$\chi D X^n F^{5-n} \tag{4.4}$$

in $10-n$ dimensions, coupling to all n scalars in the SYM multiplet in $10-n$ dimensions. This emerges from the D-brane action from the coupling to an n-form $C_{\mu_1 \ldots \mu_n}$. SYM in 10-n dimensions on $\mathbb{R} \times T^{9-n}$ is related to M-theory compactified to $n+2$ dimensions, and an n-form gauge field is dual to a scalar in $n+2$ dimensions. Such a coupling could have arisen for $d = 9 - n$ with $n = 1, 2, 3, 4, 5$, but it was only for $d = 6$ that the group theory demanded a singlet of $SL(d)$, and in that case it was the only candidate.

For the case $d = 7$, two possible couplings of the SYM to a 6-form were proposed, and in each case the 6-form couples to a 5-brane inside the 7-brane. For the coupling $C_6 \wedge F$, the 5-brane arises from a magnetic monopole configuration, as in [27], in which $\int F$ is non-zero over the T^2 transverse to the 5-brane. For the coupling $\omega_6 \wedge DX \wedge DX$, the 5-brane emerges from configurations in which $DX \wedge DX$ is the volume form for the T^2 transverse to the 5-brane, e.g. with X^α taking values in the $U(1)$ subgroup of $U(N)$ (generated by T) and identified with the coordinates on T^2, $X^\alpha = x^\alpha T$.

5. Conclusions

We have used U-duality to obtain the couplings of the SYM low-energy effective action of the matrix theory to the moduli for M-theory compactified on T^d, and considered some of the BPS branes that occur. For $d > 3$, there are wrapped BPS $d-4$ branes corresponding to YM instantons, and the U-duality symmetry mixes these with momentum modes. For $d = 4$, the 0-branes constitute an extra component of momentum and there is an extra hidden dimension that becomes manifest at finite coupling; the

U-duality results from diffeomorphisms in this higher dimensional $(5+1)$ space. For $d = 5$, the U-duality $SO(5,5;\mathbb{Z})$ is a T-duality for the 5+1 dimensional string theory, relating string winding modes and momentum modes. For $d > 5$, there are $d-4$ branes that reduce to the $d = 5$ strings in a suitable limit, and so the U-duality must relate momentum modes and $d-4$ brane wrapping modes. However, there are other backgrounds fields besides the torus metric G_{ij} and the $d-3$ form C which couples to the $d-4$ branes, and all of these are mixed by U-duality. As a result, the U-duality mixes the momentum modes, the $d-4$ brane wrapping modes and the modes coupling to these extra moduli, which are extra brane wrapping modes.

For $d = 6$, there is one extra scalar modulus χ, which couples to instantonic branes, which should play a similar role here to the instantons in 4-dimensional SYM; in both cases, there is an $SL(2,\mathbb{Z})$ acting on the coupling constant and a θ-angle, and which can be thought of formally as relating momentum modes and '-1-branes'. The $E_6(\mathbb{Z})$ is then realised as this $SL(2,\mathbb{Z})$ together with the $SL(6,\mathbb{Z})$ acting geometrically on the 6-torus, supplemented by transformations relating momentum modes (and '-1-branes') to membrane wrapping modes.

For $d = 7$, there are an extra 7 moduli corresponding to a vector or 6-form on T^7, and hence to a vector or 6-form on $T^7 \times \mathbb{R}$, and these should couple to an extra 0-brane or 5-brane. If the coupling were to an extra 0-brane, this would combine with the momentum modes in the **7** of $SL(7)$ to form an **8** of the $SL(8)$ subgroup of E_7, which would imply the existence of a hidden dimension and a formulation in 8+1 dimensions on $T^8 \times \mathbb{R}$. However, it would be hard to write down a local 8+1 dimensional theory, as it would have to couple to a metric and a self-dual 4-form on T^8, and it is not clear how to obtain such a 4-form from a field on $T^8 \times \mathbb{R}$. In fact, the D7-brane does not appear to couple to a 1-form, but does couple to a 6-form, corresponding to the fact that there are bound states of 5-branes inside 7-branes, but there appear not to be ones of 0-branes inside 7-branes (there is no vector in the type IIB theory and so no conventional 0-branes). In this case, it appears that the theory should be in 7+1 dimensions, with the $E_7(\mathbb{Z})$ mixing momentum modes with 3-brane and 5-brane wrapping modes.

For $d = 8$, there are an extra **28+8** moduli, corresponding to an extra vector or 7-form and an extra 2-form or 6-form on T^8. These would couple to an extra 0-brane or 6-brane and an extra 1-brane or 5-brane in $T^8 \times \mathbb{R}$. If the correct set of extra branes is a 0-brane plus a 5-brane, then the 0-brane combines with the momentum modes in the **8** of $SL(8)$ to form a **9** of the $SL(9)$ subgroup of E_8, which would imply the existence of a hidden dimension and a formulation in 9+1 dimensions on $T^9 \times \mathbb{R}$. The 5-brane and 4-brane in 8+1 dimensions would then arise from a 5-brane in 9+1 dimensions, and the $E_8(\mathbb{Z})$ would act as the geometrical $SL(9,\mathbb{Z})$ together with transformations mixing momentum modes with 5-brane wrapping modes. Moreover, the fact that there is $SL(9)$, not $GL(9)$, suggests that this should be a scale-invariant theory in 9+1 dimensions. Such a theory would presumably be invariant under the 10-dimensional superconformal group discussed in [31].

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For $d = 9$, the counting is more subtle, but one might expect a theory on $T^9 \times \mathbb{R}$ with at least a metric and 6-form on T^9, so that there is an extra scale in the theory compared with the $d = 8$ case, so that this is presumably not scale-invariant. Indeed, the naive counting of scalar moduli is the same as that given in the discussion of the $d = 8$ case in section 3, and gives 129 moduli, which indeed corresponds to those of $E_8/SO(16)$, plus an extra scale. (The possibility that the extra modulus corresponds to the θ angle in the coupling θF^5 would be at variance with the situation for lower d, where the scale always enters in the low-energy SYM theory.)

This suggests that the situation for $d = 8, 9$ is similar to that for $d = 4, 5$. The D4 and NS5 brane both emerge from the M5-brane, while it is conjectured that the D8 and NS9 brane both emerge from an M9-brane [23]. This suggests that the required 9+1 dimensional theories both emerge from the world-volume theories of wrapped M9-branes. A candidate for a suitable theory has been proposed in [32]; it is the target space theory of the (2,1) string.

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